The Trapezoidal Rule for Numerical Integration

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Theorem Consider $y = f(x)$ over $[x_0, x_1]$, where $x_1 = x_0 + h$. The trapezoidal rule is

$$TR(f, h) = \frac{h}{2}(f(x_0) + f(x_1)).$$

This is an numerical approximation to the integral of $f(x)$ over $[x_0, x_1]$ and we have the expression

$$\int_{x_0}^{x_1} f(x)\,dx \approx TR(f, h).$$

The remainder term for the trapezoidal rule is $R_{TR}(f, h) = -\frac{1}{12} f''(c) h^3$, where c lies somewhere between x_0 and x_1, and have the equality

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{1}{12} f''(c) h^3.$$

An intuitive method of finding the area under a curve y = f(x) is by approximating that area with a series of trapezoids that lie above the intervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$. When several trapezoids are used, we call it the composite trapezoidal rule.

Theorem (Composite Trapezoidal Rule) Consider $y = f(x)$ over $[a, b]$. Suppose that the interval $[a, b]$ is subdivided into m subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$ of equal width $h = \frac{b-a}{m}$ by using the equally spaced nodes $x_k = x_0 + k h$ for $k = 1, 2, \ldots, m$. The composite trapezoidal rule for m subintervals is

$$T(f, h) = \frac{h}{2}(f(a) + f(b)) + h \sum_{k=1}^{m} f(x_k).$$

This is an numerical approximation to the integral of $f(x)$ over $[a, b]$ and we write

$$\int_a^b f(x)\,dx \approx T(f, h).$$

Corollary (Trapezoidal Rule: Remainder term) Suppose that $[a, b]$ is subdivided into m subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$ of width $h = \frac{b-a}{m}$. The composite trapezoidal rule

$$T(f, h) = \frac{h}{2}(f(a) + f(b)) + h \sum_{k=1}^{m} f(x_k)$$

is an numerical approximation to the integral, and

$$\int_a^b f(x)\,dx = T(f, h) + E_T(f, h).$$

Furthermore, if $f(x) \in \mathbf{C}^2[a, b]$, then there exists a value c with $a < c < b$ so that the error term $E_T(f, h)$ has the form

$$E_T(f, h) = -\frac{(b-a) f^2(c)}{12} h^2.$$

This is expressed using the "big ***O***" notation $E_T(f, h) = \boldsymbol{O}(h^2)$.

Remark. When the step size is reduced by a factor of $\frac{1}{2}$ the error term $E_T(f, h)$ should be reduced by approximately $\left(\frac{1}{2}\right)^2 = 0.25$.

Algorithm Composite Trapezoidal Rule. To approximate the integral

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(f(a) + f(b)) + h \sum_{k=1}^{m-1} f(x_k),$$

by sampling $f(x)$ at the $m+1$ equally spaced points $x_k = a + k h$ for $k = 0, 1, \cdots, m$, where $h = \frac{b-a}{m}$. Notice that $x_0 = a$ and $x_m = b$.

Mathematica Subroutine (Trapezoidal Rule).

```
TrapRule[a0_, b0_, m0_] :=
  Module[{a = N[a0], b = N[b0], k, m = m0, X},
   h = (b - a)/m;
   X_k_ = a + k h;
   Return[ h/2 (f[a] + f[b]) + h Sum[f[X_k], {k, 1, m-1}] ]; ];
```

Or you can use the traditional program.

Mathematica Subroutine (Trapezoidal Rule).

```
TrapRule[a0_, b0_, m0_] :=
  Module[{a = N[a0], b = N[b0], m = m0, k},
    h = (b - a)/m;
    sum = 0;
    For[ k = 1, k ≤ m - 1, k++,
     sum = sum + f[a + h k]; ];
    Return[ h/2 (f[a] + f[b]) + h sum]; ];
```

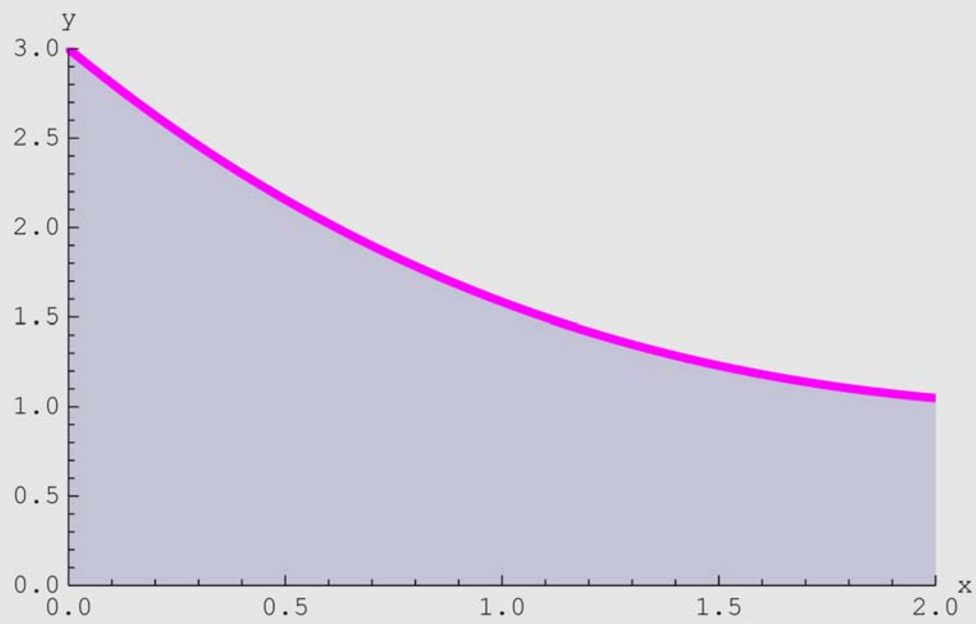
Example 1. Numerically approximate the integral $\int_0^2 \left(2 + \text{Cos}\left[2\sqrt{x}\,\right]\right) dx$ by using the trapezoidal rule with $m = 1, 2, 4, 8$, and 16 subintervals.

Solution

```
f[x_] = 2 + Cos[2 Sqrt[x]];
```

```
Needs["Graphics`FilledPlot`"]; Needs["Graphics`Colors`"]
```

```
Plot[f[x], {x, 0, 2}, PlotRange → {{0, 2}, {0, 3}},
 PlotStyle → {{Thickness[0.01`], Magenta}}, AxesLabel → {"x", "y"},
 Filling → Axis]
```



We will use simulated hand computations for the solution.

```
f[x_] = 2 + Cos[2 √x];
```

$$t1 = \frac{\frac{2-0}{1}}{2}(f[0] + f[2])$$

```
N[t1]
```

$5 + \text{Cos}\left[2\sqrt{2}\right]$

```
4.04864
```

```
t2 = ((2-0)/2)/2 (f[0] + 2 f[1] + f[2])
N[t2]
```

$$\frac{1}{2}\left(5+2\,(2+\text{Cos}[2])+\text{Cos}\left[2\sqrt{2}\right]\right)$$

3.60817

```
t4 = ((2-0)/4)/2 (f[0] + 2 f[1/2] + 2 f[1] + 2 f[3/2] + f[2])
N[t4]
```

$$\frac{1}{4}\left(5+2\,(2+\text{Cos}[2])+2\left(2+\text{Cos}\left[\sqrt{2}\right]\right)+\text{Cos}\left[2\sqrt{2}\right]+2\left(2+\text{Cos}\left[\sqrt{6}\right]\right)\right)$$

3.4971

```
t8 =
  ((2-0)/8)/2 (f[0] + 2 f[1/4] + 2 f[1/2] + 2 f[3/4] + 2 f[1] + 2 f[5/4] + 2 f[3/2] + 2 f[7/4] + f[2])
N[t8]
```

$$\frac{1}{8}\left(5+2\,(2+\text{Cos}[1])+2\,(2+\text{Cos}[2])+2\left(2+\text{Cos}\left[\sqrt{2}\right]\right)+\text{Cos}\left[2\sqrt{2}\right]+2\left(2+\text{Cos}\left[\sqrt{3}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{5}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{6}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{7}\right]\right)\right)$$

3.46928

```
t16 =
```

$$\frac{\frac{2-0}{16}}{2}\left(f[0]+2f\left[\frac{1}{8}\right]+2f\left[\frac{1}{4}\right]+2f\left[\frac{3}{8}\right]+2f\left[\frac{1}{2}\right]+2f\left[\frac{5}{8}\right]+2f\left[\frac{3}{4}\right]+2f\left[\frac{7}{8}\right]+2f[1]+2f\left[\frac{9}{8}\right]+2f\left[\frac{5}{4}\right]+2f\left[\frac{11}{8}\right]+2f\left[\frac{3}{2}\right]+2f\left[\frac{13}{8}\right]+2f\left[\frac{7}{4}\right]+2f\left[\frac{15}{8}\right]+f[2]\right)$$

```
N[t16]
```

$$\frac{1}{16}\left(5+2\,(2+\text{Cos}[1])+2\,(2+\text{Cos}[2])+2\left(2+\text{Cos}\left[\sqrt{\frac{3}{2}}\right]\right)+2\left(2+\text{Cos}\left[\frac{1}{\sqrt{2}}\right]\right)+2\left(2+\text{Cos}\left[\frac{3}{\sqrt{2}}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{2}\right]\right)+\text{Cos}\left[2\sqrt{2}\right]+2\left(2+\text{Cos}\left[\sqrt{\frac{5}{2}}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{3}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{\frac{7}{2}}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{5}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{\frac{11}{2}}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{6}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{\frac{13}{2}}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{7}\right]\right)+2\left(2+\text{Cos}\left[\sqrt{\frac{15}{2}}\right]\right)\right)$$

```
3.46232
```

Example 2. Numerically approximate the integral $\int_0^2 \left(2+\text{Cos}\left[2\sqrt{x}\right]\right) dx$ by using the trapezoidal rule with m = 50, 100, 200, 400 and 800 subintervals.

Solution

We will use the subroutine for the solution.

$$f[x\_] = 2+\text{Cos}\left[2\sqrt{x}\right];$$

```
t50 = TrapRule[0, 2, 50]
NumberForm[t50, 12]
```

```
3.46024
```

```
3.46023529269
```

```
t100 = TrapRule[0, 2, 100]
NumberForm[t100, 12]
```

```
3.46006
```

```
3.46005707746
```

```
t200 = TrapRule[0, 2, 200]
NumberForm[t200, 12]
```

```
3.46001
```

```
3.4600125235
```

```
t400 = TrapRule[0, 2, 400]
NumberForm[t400, 12]
```

```
3.46
```

```
3.460001385
```

```
t800 = TrapRule[0, 2, 800]
NumberForm[t800, 12]
```

```
3.46
```

```
3.45999860038
```

Example 3. Find the analytic value of the integral

$\int_0^2 \left(2 + \text{Cos}\left[2\sqrt{x}\right]\right) dx$ (i.e. find the "true value").

Solution

```
val = ∫_0^2 (2 + Cos[2 √x]) dx
```

$\frac{1}{2}\left(7 + \text{Cos}\left[2\sqrt{2}\right] + 2\sqrt{2}\,\text{Sin}\left[2\sqrt{2}\right]\right)$

```
N[val]
```

```
3.46
```

```
NumberForm[N[val] , 12]
```

```
3.45999767217
```

Example 4. Use the "true value" in example 3 and find the error for the trapezoidal rule approximations in example 2.

Solution

```
val - t50
```

```
-0.000237621
```

```
val - t100
```

```
-0.0000594053
```

```
val - t200
```

```
-0.0000148513
```

```
val - t400
```

-3.71283×10^{-6}

```
val - t800
```

-9.28209×10^{-7}

Example 5. When the step size is reduced by a factor of $\frac{1}{2}$ the error term $E_T(f, h)$ should be reduced by approximately $\left(\frac{1}{2}\right)^2 = 0.25$. Explore this phenomenon.

Solution

$$\frac{\texttt{val - t100}}{\texttt{val - t50}}$$

0.250001

$$\frac{\texttt{val - t200}}{\texttt{val - t100}}$$

0.25

$$\frac{\texttt{val - t400}}{\texttt{val - t200}}$$

0.25

$$\frac{\texttt{val - t800}}{\texttt{val - t400}}$$

0.25

Example 6. Numerically approximate the integral

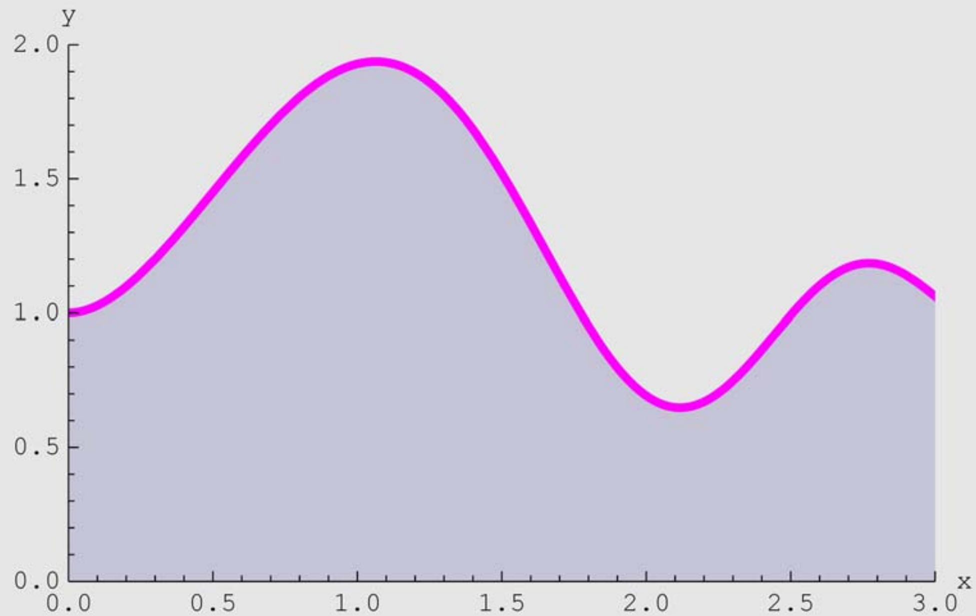
$\int_0^3 (3 e^{-x} \text{Sin}[x^2] + 1) \, dx$ by using the trapezoidal rule with m = 1, 2, 4, 8, and 16 subintervals.

Solution

```
f[x_] = 3 e^-x Sin[x^2] + 1;
```

```
Needs["Graphics`FilledPlot`"]; Needs["Graphics`Colors`"];
```

```
Plot[f[x], {x, 0, 3}, PlotRange → {{0, 3}, {0, 2}},
 PlotStyle → {{Thickness[0.01`], Magenta}}, AxesLabel → {"x", "y"},
 Filling → Axis]
Print["f[x] = ", f[x]];
```



$f[x] = 1 + 3 e^{-x} \text{Sin}[x^2]$

We will use simulated hand computations for the solution.

$$t1 = \frac{\frac{3-0}{1}}{2}\,(f[0] + f[3])$$

N[t1]

$$\frac{3}{2}\left(2 + \frac{3\,\mathrm{Sin}[9]}{e^3}\right)$$

3.09233

$$t2 = \frac{\frac{3-0}{2}}{2}\left(f[0] + 2\,f\left[\frac{3}{2}\right] + f[3]\right)$$

N[t2]

$$\frac{3}{4}\left(2 + 2\left(1 + \frac{3\,\mathrm{Sin}\left[\frac{9}{4}\right]}{e^{3/2}}\right) + \frac{3\,\mathrm{Sin}[9]}{e^3}\right)$$

3.82742

$$t4 = \frac{\frac{3-0}{4}}{2}\left(f[0] + 2\,f\left[\frac{3}{4}\right] + 2\,f\left[\frac{3}{2}\right] + 2\,f\left[\frac{9}{4}\right] + f[3]\right)$$

N[t4]

$$\frac{3}{8}\left(2 + 2\left(1 + \frac{3\,\mathrm{Sin}\left[\frac{9}{16}\right]}{e^{3/4}}\right) + 2\left(1 + \frac{3\,\mathrm{Sin}\left[\frac{9}{4}\right]}{e^{3/2}}\right) + 2\left(1 + \frac{3\,\mathrm{Sin}\left[\frac{81}{16}\right]}{e^{9/4}}\right) + \frac{3\,\mathrm{Sin}[9]}{e^3}\right)$$

3.75775

```
t8 =
```

$$\frac{\frac{3-0}{8}}{2}\left(f[0]+2f\left[\frac{3}{8}\right]+2f\left[\frac{3}{4}\right]+2f\left[\frac{9}{8}\right]+2f\left[\frac{3}{2}\right]+2f\left[\frac{15}{8}\right]+2f\left[\frac{9}{4}\right]+2f\left[\frac{21}{8}\right]+f[3]\right)$$

```
N[t8]
```

$$\frac{3}{16}\left(2+2\left(1+\frac{3\,\text{Sin}\left[\frac{9}{64}\right]}{e^{3/8}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{9}{16}\right]}{e^{3/4}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{81}{64}\right]}{e^{9/8}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{9}{4}\right]}{e^{3/2}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{225}{64}\right]}{e^{15/8}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{81}{16}\right]}{e^{9/4}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{441}{64}\right]}{e^{21/8}}\right)+\frac{3\,\text{Sin}[9]}{e^{3}}\right)$$

```
3.81909
```

```
t16 =
 (3-0)/16/2 (f[0] + 2 f[3/16] + 2 f[3/8] + 2 f[9/16] + 2 f[3/4] + 2 f[15/16] + 2 f[9/8] +
    2 f[21/16] + 2 f[3/2] + 2 f[27/16] + 2 f[15/8] + 2 f[33/16] + 2 f[9/4] + 2 f[39/16] +
    2 f[21/8] + 2 f[45/16] + f[3])
N[t16]
```

$$\frac{3}{32}\left(2+2\left(1+\frac{3\,\text{Sin}\left[\frac{9}{256}\right]}{e^{3/16}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{9}{64}\right]}{e^{3/8}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{81}{256}\right]}{e^{9/16}}\right)+\right.$$
$$2\left(1+\frac{3\,\text{Sin}\left[\frac{9}{16}\right]}{e^{3/4}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{225}{256}\right]}{e^{15/16}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{81}{64}\right]}{e^{9/8}}\right)+$$
$$2\left(1+\frac{3\,\text{Sin}\left[\frac{441}{256}\right]}{e^{21/16}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{9}{4}\right]}{e^{3/2}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{729}{256}\right]}{e^{27/16}}\right)+$$
$$2\left(1+\frac{3\,\text{Sin}\left[\frac{225}{64}\right]}{e^{15/8}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{1089}{256}\right]}{e^{33/16}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{81}{16}\right]}{e^{9/4}}\right)+$$
$$\left.2\left(1+\frac{3\,\text{Sin}\left[\frac{1521}{256}\right]}{e^{39/16}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{441}{64}\right]}{e^{21/8}}\right)+2\left(1+\frac{3\,\text{Sin}\left[\frac{2025}{256}\right]}{e^{45/16}}\right)+\frac{3\,\text{Sin}[9]}{e^{3}}\right)$$

```
3.82821
```

Example 7. Numerically approximate the integral $\int_0^3 \left(3\, e^{-x}\, \text{Sin}\left[x^2\right]+1\right) dx$ by using the trapezoidal rule with m = 50, 100, 200, 400 and 800 subintervals.

Solution

We will use the subroutine for the solution.

```
f[x_] = 3 e^-x Sin[x^2] + 1;
```

```
t50 = TrapRule[0, 3, 50]
NumberForm[t50, 12]
```

```
3.8306
```

```
3.83060406834
```

```
t100 = TrapRule[0, 3, 100]
NumberForm[t100, 12]
```

```
3.8308
```

```
3.83080248387
```

```
t200 = TrapRule[0, 3, 200]
NumberForm[t200, 12]
```

```
3.83085
```

```
3.83085192875
```

```
t400 = TrapRule[0, 3, 400]
NumberForm[t400, 12]
```

```
3.83086
```

```
3.83086428005
```

```
t800 = TrapRule[0, 3, 800]
NumberForm[t800, 12]
```

```
3.83087
```

```
3.83086736725
```

Example 8. Find the analytic value of the integral

$\int_0^3 (3\, e^{-x} \operatorname{Sin}[x^2] + 1)\, dx$ (i.e. find the "true value").

Solution

```
val = ∫_0^3 (3 e^-x Sin[x^2] + 1) dx
```

$$3 + \frac{3}{4} (-1)^{1/4} e^{-\frac{i}{4}} \sqrt{\pi} \left(\text{Erf}\left[\left(3 - \frac{i}{2}\right) (-1)^{1/4}\right] + \text{Erf}\left[\frac{1}{2} (-1)^{3/4}\right] + i\, e^{\frac{i}{2}} \left(\text{Erf}\left[\frac{1}{2} (-1)^{1/4}\right] + \text{Erf}\left[\left(3 + \frac{i}{2}\right) (-1)^{3/4}\right]\right)\right)$$

```
val = N[Re[val]]
```

```
3.83087
```

```
NumberForm[val, 12]
```

```
3.83086839627
```

Example 9. Use the "true value" in example 8 and find the error for the trapezoidal rule approximations in exercise 7.

Solution

```
val - t50
```

```
0.000264328
```

```
val - t100
```

```
0.0000659124
```

```
val - t200
```

```
0.0000164675
```

```
val - t400
```

4.11622×10^{-6}

```
val - t800
```

1.02901×10^{-6}

Example 10. When the step size is reduced by a factor of $\frac{1}{2}$ the error term $E_T(f, h)$ should be reduced by approximately $\left(\frac{1}{2}\right)^2 = 0.25$. Explore this phenomenon.

Solution

$$\frac{\text{val} - \text{t100}}{\text{val} - \text{t50}}$$

0.249358

$$\frac{\text{val} - \text{t200}}{\text{val} - \text{t100}}$$

0.249839

$$\frac{\text{val} - \text{t400}}{\text{val} - \text{t200}}$$

0.24996

$$\frac{\text{val} - \text{t800}}{\text{val} - \text{t400}}$$

0.24999

Recursive Integration Rules

Theorem (Successive Trapezoidal Rules) Suppose that $j \geq 1$ and the points $\{x_k = a + k\,h\}$ subdivide $[a, b]$ into $2^j = 2\,m$ subintervals equal width $h = \frac{b-a}{2^j}$. The trapezoidal rules $T(f, h)$ and $T(f, 2\,h)$ obey the relationship

$$T(f, h) = \frac{T(f, 2\,h)}{2} + h \sum_{k=1}^{m} f(x_{2k-1}).$$

Definition (Sequence of Trapezoidal Rules) Define $T(0) = \frac{h}{2}(f(a) + f(b))$, which is the trapezoidal rule with step size $h = b - a$. Then for each $j \geq 1$ define $T(0) = T(f, h)$, where $T(f, h)$ is the trapezoidal rule with step size $h = \frac{b-a}{2^j}$.

Corollary (Recursive Trapezoidal Rule) Start with $T(0) = \frac{h}{2}(f(a) + f(b))$. Then a sequence of trapezoidal rules $\{T(j)\}$ is generated by the recursive formula

$$T(j) = \frac{T(j-1)}{2} + h \sum_{k=1}^{m} f(x_{2k-1}) \quad \text{for} \quad j = 1, 2, \ldots .$$

where $h = \frac{b-a}{2^j}$ and $\{x_k = a + k\,h\}$.

The recursive trapezoidal rule is used for the Romberg integration algorithm.

Example 11. Let $f[x] = 1 + e^{-x} \text{Sin}\left[8\,x^{2/3}\right]$ over $[0, 2]$. Use the Trapezoidal Rule to approximate the value of the integral.

Solution

```
f[x_] = 1 + e^-x Sin[8 x^(2/3)];
 (a = 0.`;) (b = 2.`;)
  (M = {1, 2, 4, 6, 8, 10, 12, 14, 16, 20, 24, 28, 32, 40, 50, 60, 70,
      80, 100, 120};) (tbl = Table[j, {j, 1, 20}];) (Print[""];)
  (For[j = 1, j ≤ 20, j++, tbl⟦j⟧ = {M⟦j⟧ + 1, TrapRule[a, b, M⟦j⟧]};];)
 Print[tbl];
Needs["Graphics`Colors`"];
Plot[f[x], {x, 0, 2}, PlotRange → {{0, 2.01`}, {0, 2.01`}},
  Ticks → {Range[0, 2, 0.5`], Range[0, 2, 0.5`]}, Axes → True,
  PlotStyle → Magenta]
```

```
Null^6
```

```
{{2, 2.01792}, {3, 2.37293}, {5, 1.80207}, {7, 1.85519},
 {9, 1.90659}, {11, 1.93776}, {13, 1.95723}, {15, 1.97011},
 {17, 1.97906}, {21, 1.9904}, {25, 1.99709}, {29, 2.00139},
 {33, 2.00433}, {41, 2.00802}, {51, 2.01057}, {61, 2.01206},
 {71, 2.01301}, {81, 2.01366}, {101, 2.01447}, {121, 2.01494}}
```